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Remarks on quantization of Pais–Uhlenbeck oscillators

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Abstract
This work is concerned with a quantization of the Pais–Uhlenbeck oscillators from the point of view of their multi-Hamiltonian structures. It is shown that the $2n$th-order oscillator with a simple spectrum can be quantized as the usual anisotropic n-dimensional oscillator.

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1. Introduction

This work is concerned with a quantization of the Pais–Uhlenbeck oscillators from the point of view of their multi-Hamiltonian structures. The family of such oscillators was introduced in the paper [1] as a toy model to study field theories with higher derivative terms. Evolution of the Pais–Uhlenbeck $2n$th-order oscillator is defined by the following equation

$$\prod_{i=1}^{n}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega_i^2\right)x=0 \tag{1}$$

where $\Omega=(\omega_i)$, $i=1,\ldots,n$ is a set of positive parameters (frequencies). Equation (1) can be obtained by variation of the Lagrangian

$$L=-x\left(\prod_{i=1}^{n}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega_i^2\right)\right)x. \tag{2}$$

Introducing the natural oscillator coordinates

$$q_k=\prod_{i=1}^{k-1}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega_i^2\right)\prod_{i=k+1}^{n}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega_i^2\right)x \tag{3}$$

and conjugate momenta Pais and Uhlenbeck proved that the Hamiltonian related to the Lagrangian (2) has the form

$$H_{\mathrm{PU}} = \frac{1}{2}\sum_{i=1}^{n}(-1)^{i-1}\left(p_i^2 + \omega_i^2 q_i^2\right). \tag{4}$$

It follows from this expression that the Hamiltonian is not positive defined. Recently, the quantization of the fourth-order Pais–Uhlenbeck oscillator was carried out in some detail by Mannheim and Davidson [2] (see also [3]). The starting point for the authors of [2] was the Lagrangian (2) with $n = 2$. Using the Dirac method they obtained the Hamiltonian (see below first expression from (12)) which is different in its form from the Pais–Uhlenbeck one (4). But the quantization of this new Hamiltonian leads to the Hilbert space $\mathcal{H}$ containing negative norm states (this is because of improper sign of the commutator of a pair of creation and annihilation operators). This result is treated usually as the intrinsic property of theories with higher derivative terms.

However, there are arguments that a satisfactory quantization of the Pais–Uhlenbeck oscillators can be carried out. For simplicity, let us consider the fourth-order oscillator. In terms of the oscillator coordinates (3)

$$q_1 = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_2^2 x, \qquad q_2 = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_1^2 x,$$

and momenta (or velocities, which are identical to the momenta in this case)

$$p_1 = \frac{\mathrm{d}q_1}{\mathrm{d}t}, \qquad p_2 = \frac{\mathrm{d}q_2}{\mathrm{d}t}$$

equation (1) can be rewritten as a canonical system of Hamiltonian equations of motion for the two-dimensional anisotropic oscillator

$$\begin{aligned} \frac{\mathrm{d}q_1}{\mathrm{d}t} &= \frac{\partial H_C}{\partial p_1} = p_1, & \frac{\mathrm{d}p_1}{\mathrm{d}t} &= -\frac{\partial H_C}{\partial q_1} = -\omega_1^2 q_1, \\ \frac{\mathrm{d}q_2}{\mathrm{d}t} &= \frac{\partial H_C}{\partial p_2} = p_2, & \frac{\mathrm{d}p_2}{\mathrm{d}t} &= -\frac{\partial H_C}{\partial p_2} = -\omega_1^2 q_2 \end{aligned} \tag{5}$$

where

$$H_C = \tfrac{1}{2}\left(p_1^2 + \omega_1^2 q_1^2\right) + \tfrac{1}{2}\left(p_2^2 + \omega_2^2 q_2^2\right). \tag{6}$$

The canonical quantization of this oscillator leads to the usual commutation relations among the creation and annihilation operators and to the standard Hilbert space with a positive norm. Thus, it seems natural to deduce a relevant (for a quantization) Hamiltonian formulation of the model directly from the equation of motion (1) omitting a Lagrangian formulation.

The difference of the Hamiltonians (4) (for $n = 2$) and (6) tells us that the Pais–Uhlenbeck oscillator equation of motion can be obtained using different Hamiltonian structures. In another words this oscillator is a bi-Hamiltonian system [4, 5]. In the case of the classical fourth-order oscillator this fact was established in [6].

In the present paper we consider the quantization of the Pais–Uhlenbeck oscillators from the point of view of their multi-Hamiltonian nature. We show that the $2n$th-order Pais–Uhlenbeck oscillator with a simple spectrum (all frequencies from the set Ω are different) can be quantized as the usual anisotropic n-dimensional oscillator since in the terms of the standard creation and annihilation operators the Hamiltonians of these oscillators coincide. We start from the case of the fourth-order oscillator and then present the main formulae for the general case.

2. The fourth-order oscillator

The equation of motion of the fourth-order Pais–Uhlenbeck oscillator ($n = 2$ in (1)) has the form

$$\frac{d^4x}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2x}{dt^2} + \omega_1^2\omega_2^2 x = 0. \tag{7}$$

It can be written in the form of a system of first-order equations

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = x_3, \qquad \frac{dx_3}{dt} = x_4, \qquad \frac{dx_4}{dt} = -(\omega_1^2 + \omega_2^2)x_3 - \omega_1^2\omega_2^2 x_1, \tag{8}$$

where x_i, $i = 1, \ldots, 4$, are local coordinates of the 'phase' space ($x_1 = x$ is the coordinate in the original space, x_2 is the velocity and so on). Integral curves of the vector field

$$\mathbf{V} = x_2\partial_1 + x_3\partial_2 + x_4\partial_3 - \left((\omega_1^2 + \omega_2^2)x_3 + \omega_1^2\omega_2^2 x_1\right)\partial_4, \qquad \partial_i = \frac{\partial}{\partial x_i}, \tag{9}$$

are exactly solutions of the system (8) (see, for instance [7, 8]). The simplest way to obtain integrals of motion of the oscillator is to use the following equation:

$$\mathbf{V}(H) = 0. \tag{10}$$

Note here that for this purpose in the paper [6] was applied the known solution of equation (7). In the local coordinates equation (10) has the form

$$x_2\partial_1 H + x_3\partial_2 H + x_4\partial_3 H - \left((\omega_1^2 + \omega_2^2)x_3 + \omega_1^2\omega_2^2 x_1\right)\partial_4 H = 0.$$

Since the components of the field $\mathbf{V}$ are homogeneous linear coordinate functions then the analytic solutions of equation (10) are homogeneous polynomials in x_i. For the considered Pais–Uhlenbeck oscillator we have two independent positive defined quadratic integrals of motion

$$\begin{aligned} H_1 &= \tfrac{1}{2}\left(x_4 + \omega_2^2 x_2\right)^2 + \tfrac{1}{2}\omega_1^2\left(x_3 + \omega_2^2 x_1\right)^2, \\ H_2 &= \tfrac{1}{2}\left(x_4 + \omega_1^2 x_2\right)^2 + \tfrac{1}{2}\omega_2^2\left(x_3 + \omega_1^2 x_1\right)^2. \end{aligned} \tag{11}$$

Taking into account the above definition of the oscillator coordinates q_i and related momenta p_i one can see that the Hamiltonian (6) is the sum of integrals H_1 and H_2 from (11)

$$H_C = H_1 + H_2,$$

whereas the Pais–Uhlenbeck Hamiltonian is their difference

$$H_{\mathrm{PU}} = H_1 - H_2.$$

Let us point out that the coordinates q_1, q_2 and the integrals (11) are degenerate in the case $\omega_1 = \omega_2$. The simplest linear combinations of H_1 and H_2

$$\begin{aligned} C_1 &= \frac{\omega_1^2 H_1 - \omega_2^2 H_2}{\omega_1^2 - \omega_2^2} = -\frac{1}{2}\omega_1^2\omega_2^2 x_2^2 + \frac{1}{2}(\omega_1^2 + \omega_2^2)x_3^2 + \frac{1}{2}x_4^2 + \omega_1^2\omega_2^2 x_1 x_3, \\ C_2 &= -\frac{H_1 - H_2}{\omega_1^2 - \omega_2^2} = \frac{1}{2}\omega_1^2\omega_2^2 x_1^2 + \frac{1}{2}(\omega_1^2 + \omega_2^2)x_2^2 - \frac{1}{2}x_3^2 + x_2 x_4, \end{aligned} \tag{12}$$

gives us the pair of integrals which are distinct at $\omega_1 = \omega_2$, but C_1 and C_2 are not positive defined. The integral C_1 was obtained in [2] and it was taken as a Hamiltonian of the fourth-order Pais–Uhlenbeck oscillator. The notation of [2] and ours are related by the following formulae:

$$x_1 = p_q, \qquad x_2 = \gamma\omega_1^2\omega_2^2 q, \qquad x_3 = \gamma\omega_1^2\omega_2^2 x, \qquad x_4 = \omega_1^2\omega_2^2 p_x.$$

Now introduce a Poisson structure for the considered oscillator. Recall that a Poisson structure on a manifold is defined by a rank-2 contravariant tensor field Π which is skew symmetric

$$\Pi^{ij} = -\Pi^{ji}$$

and satisfies the condition

$$[[\Pi, \Pi]]^{ijk} := \sum_m (\Pi^{mi}\partial_m \Pi^{jk} + \Pi^{mj}\partial_m \Pi^{ki} + \Pi^{mk}\partial_m \Pi^{ij}) = 0.$$

Any Poisson structure induces a Poisson brackets $\{F, G\}$ of arbitrary differentiable functions F, G on a manifold M. In local coordinates x_i the brackets are defined by the formula

$$\{F, G\} = \Pi^{ij}\partial_i F \partial_j G.$$

The vector field $\mathbf{V}$ is called locally Hamiltonian if there is a Poisson structure Π such that

$$\mathcal{L}_{\mathbf{V}}(\Pi) = 0, \tag{13}$$

where $\mathcal{L}_{\mathbf{V}}$ defines the Lie derivative along the field $\mathbf{V}$. In local coordinates the above relation has the form

$$V^k \frac{\partial \Pi^{ij}}{\partial x^k} - \frac{\partial V^i}{\partial x^k}\Pi^{kj} - \Pi^{ik}\frac{\partial V^j}{\partial x^k} = 0,$$

where V^i and Π^{ij} are components of the vector field $\mathbf{V}$ and the Poisson tensor Π. If there is such a differentiable function H that $\mathbf{V}$ can be represented in the form

$$\mathbf{V}_H(.) = \{., H\},$$

then H is called a Hamiltonian. In this case equations of motion take the canonical form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \{x_i, H\}. \tag{14}$$

Considering the relation (13) as an equation one can obtain a Poisson tensor Π related to the field $\mathbf{V}$. The simplest solution of this equation is a two-parameter nondegenerate Poisson tensor with constant components. Its components are represented in the following table (f and g are arbitrary parameters)

$$[\Pi^{ij}_{f,g}] = \begin{bmatrix} 0 & f & 0 & g \\ -f & 0 & -g & 0 \\ 0 & g & 0 & -\omega_1^2\omega_2^2 f - \left(\omega_1^2 + \omega_2^2\right)g \\ -g & 0 & \omega_1^2\omega_2^2 f + \left(\omega_1^2 + \omega_2^2\right)g & 0 \end{bmatrix}. \tag{15}$$

This Poisson tensor $\Pi_{f,g}$ is obviously skew symmetric and satisfy the condition $[[\Pi, \Pi]]^{ijk} = 0$ in view of its constancy. It induces the following Poisson brackets for the coordinate functions:

$$\begin{aligned} &\{x_1, x_2\}_{f,g} = f, && \{x_1, x_4\}_{f,g} = g, \\ &\{x_2, x_3\}_{f,g} = -g, && \{x_3, x_4\}_{f,g} = -\omega_1^2\omega_2^2 f - \left(\omega_1^2 + \omega_2^2\right)g. \end{aligned} \tag{16}$$

It is not difficult to check that the dynamical equations (8) are generated by these brackets

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \{x_i, H\}_{f,g}$$

together with the Hamiltonian function

$$H = a_1 H_1 + a_2 H_2, \tag{17}$$

where the coefficients a_i have the form

$$a_1 = \frac{1}{(\omega_2^2 - \omega_1^2)(\omega_2^2 f + g)}, \qquad a_2 = -\frac{1}{(\omega_2^2 - \omega_1^2)(\omega_1^2 f + g)}, \tag{18}$$

and can be chosen positive. Thus the dynamical equations (8) (and the field $\mathbf{V}$ itself) are Hamiltonian ones and the two-parameter function H plays the role of a Hamiltonian. We remark that the integrals of motion C_1 and C_2 are in involution with respect to these brackets

$$\{C_1, C_2\}_{f,g} = 0.$$

In the classical case the parameters f and g can be taken either arbitrary or fixed in any appropriate manner. For instance, we can put

$$f = -\frac{1}{\omega_1^2\omega_2^2}, \qquad g = 0.$$

This choice gives the following nonzero Poisson brackets of the coordinate function x_i:

$$\{x_1, x_2\}_1 = -\frac{1}{\omega_1^2\omega_2^2}, \qquad \{x_3, x_4\}_1 = 1. \tag{19}$$

Other simple choice

$$f = 0, \qquad g = 1,$$

gives the brackets

$$\{x_1, x_4\}_2 = 1, \qquad \{x_2, x_3\}_2 = -1, \qquad \{x_3, x_4\}_2 = -(\omega_1^2 + \omega_2^2). \tag{20}$$

Both mentioned Poisson brackets generate the dynamical equations (8)

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \{x_i, C_1\}_1 = \{x_i, C_2\}_2. \tag{21}$$

Thus the fourth-order Pais–Uhlenbeck oscillator is a bi-Hamiltonian system ([4, 5, 8]) with the Hamiltonians C_1, C_2 and the Poisson structures Π_1, Π_2. First from these structures has been used in [2]. Let us note in conclusion that there is no such a constant Poisson structure which generates the equations (8) together with any of the Hamiltonians (11).

The semiclassical quantization of the Poisson structure (15) can be considered as exact because of the dynamical equations (8) of the fourth-order Pais–Uhlenbeck oscillator are reducible to the canonically quantized form (5) by the linear transformation. Assume that the Hermitian operators $\hat{x}_i, i = 1, \ldots, 4$, related to the dynamical variables of the classical system x_i, subject to the following commutation relations:

$$\begin{aligned} &[\hat{x}_1, \hat{x}_2]_{f,g} = \mathrm{i}\hbar f, && [\hat{x}_1, \hat{x}_4]_{f,g} = \mathrm{i}\hbar g, \\ &[\hat{x}_2, \hat{x}_3]_{f,g} = -\mathrm{i}\hbar g, && [\hat{x}_3, \hat{x}_4]_{f,g} = -\mathrm{i}\hbar\omega_1^2\omega_2^2 f - \mathrm{i}\hbar(\omega_1^2 + \omega_2^2)g. \end{aligned} \tag{22}$$

Using these relations it is not difficult to show that the quantum dynamical equation

$$\frac{\mathrm{d}\hat{x}_1}{\mathrm{d}t} = \hat{x}_2, \qquad \frac{\mathrm{d}\hat{x}_2}{\mathrm{d}t} = \hat{x}_3, \qquad \frac{\mathrm{d}\hat{x}_3}{\mathrm{d}t} = \hat{x}_4, \qquad \frac{\mathrm{d}\hat{x}_4}{\mathrm{d}t} = -(\omega_1^2 + \omega_2^2)\hat{x}_3 - \omega_1^2\omega_2^2\hat{x}_1 \tag{23}$$

can be represented in the Heisenberg form

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} = [\hat{x}_i, \hat{H}]_{f,g} \tag{24}$$

where the quantum Hamiltonian $\hat{H}$ is obtained from the classical one (17) by the replacement of the classical dynamical variables by the quantum dynamical variables $x_i \to \hat{x}_i, i = 1, \ldots, 4$. We remark that there are no problems with the ordering of the quantum variables because all the terms of the form $\hat{x}_i\hat{x}_j$ in $\hat{H}$ include only commutative operators. It is easy to check that the quantum analogues of all the above-considered classical integrals of motion $\hat{H}_1, \hat{H}_2, \hat{C}_1, \hat{C}_2$ commutate with each other. As in the classical case, fixing the parameters f and g one can obtain independent realizations of the quantum dynamical equations in the Heisenberg form with different Hamiltonians. For example, putting $f = -\frac{1}{\omega_1^2\omega_2^2}$, $g = 0$ (see (19)) or $f = 0$, $g = 1$, (see (20)), we obtain the equations

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} = [\hat{x}_i, \hat{C}_1]_1, \qquad \mathrm{i}\hbar\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} = [\hat{x}_i, \hat{C}_2]_2$$

respectively. In these equations for the calculation of the commutator $[.\,,.]_1$ it is necessary to use the first fixed pair f, g, and for the calculation of the commutator $[.\,,.]_2$ it is necessary to use the second one. In both cases the roles of Hamiltonians play the operators

$$\begin{aligned} \hat{C}_1 &= -\tfrac{1}{2}\omega_1^2\omega_2^2\hat{x}_2^2 + \tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)\hat{x}_3^2 + \tfrac{1}{2}\hat{x}_4^2 + \omega_1^2\omega_2^2\hat{x}_1\hat{x}_3, \\ \hat{C}_2 &= \tfrac{1}{2}\omega_1^2\omega_2^2\hat{x}_1^2 + \tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)\hat{x}_2^2 - \tfrac{1}{2}x_3^2 + \hat{x}_2\hat{x}_4. \end{aligned} \tag{25}$$

Thus, we obtain that the quantum version of the fourth-order Pais–Uhlenbeck oscillator is the bi-Hamiltonian system as well.

In view of the linearity of the quantum dynamical equations (23) one can easily write out their operator solution

$$\begin{aligned} \hat{x}_1 &= \mathrm{e}^{-\mathrm{i}\omega_1 t}a_1 + a_2\,\mathrm{e}^{-\mathrm{i}\omega_2 t}a_2 + \text{h.c.}, & \hat{x}_2 &= -\mathrm{i}\omega_1\,\mathrm{e}^{-\mathrm{i}\omega_1 t}a_1 - \mathrm{i}\omega_2\,\mathrm{e}^{-\mathrm{i}\omega_2 t}a_2 + \text{h.c.}, \\ \hat{x}_3 &= -\omega_1^2\,\mathrm{e}^{-\mathrm{i}\omega_1 t}a_1 - \omega_2^2\,\mathrm{e}^{-\mathrm{i}\omega_2 t}a_2 + \text{h.c.}, & \hat{x}_4 &= \mathrm{i}\omega_1^3\,\mathrm{e}^{-\mathrm{i}\omega_1 t}a_1 + \mathrm{i}\omega_2^3\,\mathrm{e}^{-\mathrm{i}\omega_2 t}a_2 + \text{h.c.}, \end{aligned} \tag{26}$$

where we took into account the self-conjugacy of the dynamical variables $\hat{x}_i$. Using the commutation relations (22) we obtain nonzero commutators among the operators a_i, a_i^+, $i = 1, 2$

$$\left[a_1, a_1^+\right] = \frac{\hbar\left(\omega_2^2 f + g\right)}{2\omega_1\left(\omega_2^2-\omega_1^2\right)}, \qquad \left[a_2, a_2^+\right] = -\frac{\hbar\left(\omega_1^2 f + g\right)}{2\omega_2\left(\omega_2^2-\omega_1^2\right)}. \tag{27}$$

From the condition

$$\left[a_1, a_1^+\right] = \left[a_2, a_2^+\right] = 1, \tag{28}$$

imposed usually on commutators of creation and annihilation operators, we uniquely fix the parameters f and g

$$f = \frac{2}{\hbar}(\omega_1+\omega_2), \qquad g = -\frac{2}{\hbar}\left(\omega_1^3+\omega_2^3\right). \tag{29}$$

Substituting solution (26) into the expressions for $\hat{H}_1, \hat{H}_2, \hat{H}, \hat{C}_1, \hat{C}_2$ and taking into account the commutation relations with defined above parameters (29) we obtain

$$\begin{aligned} \hat{H}_1 &= \hat{C}_1 + \omega_2^2\hat{C}_2 = 2\omega_1^2\left(\omega_2^2-\omega_1^2\right)^2\left(a_1^+a_1+\tfrac{1}{2}\right), \\ \hat{H}_2 &= \hat{C}_1 + \omega_1^2\hat{C}_2 = 2\omega_2^2\left(\omega_2^2-\omega_1^2\right)^2\left(a_2^+a_2+\tfrac{1}{2}\right), \end{aligned} \tag{30}$$

$$\hat{H} = a_1\hat{H}_1 + a_2\hat{H}_2 = \hbar\omega_1\left(a_1^+a_1+\tfrac{1}{2}\right) + \hbar\omega_2\left(a_2^+a_2+\tfrac{1}{2}\right) \tag{31}$$

and

$$\begin{aligned} \hat{C}_1 &= 2\left(\omega_2^2-\omega_1^2\right)\left(-\omega_1^4\left(a_1^+a_1+\tfrac{1}{2}\right) + \omega_2^4\left(a_2^+a_2+\tfrac{1}{2}\right)\right), \\ \hat{C}_2 &= 2\left(\omega_2^2-\omega_1^2\right)\left(\omega_1^2\left(a_1^+a_1+\tfrac{1}{2}\right) - \omega_2^2\left(a_2^+a_2+\tfrac{1}{2}\right)\right). \end{aligned} \tag{32}$$

These formulae show that in the case $\omega_1 \neq \omega_2$ the quantum fourth-order Pais–Uhlenbeck oscillator can be quantized as the usual two-dimensional anisotropic oscillator. Hence, using the operators a_i^+, a_i one can construct the standard Hilbert state space $\mathcal{H}$ with the positive normalized basis vectors

$$|\psi_{mn}\rangle = \frac{1}{\sqrt{m!n!}}\left(a_1^+\right)^m\left(a_2^+\right)^n|0\rangle,$$

and the vacuum vector satisfying the condition

$$a_1|0\rangle = a_2|0\rangle = 0.$$

All quantum integrals of motion $\hat{C}_i, \hat{H}_i$ are diagonal in this basis.

3. The general case

Let us return to the general case of the Pais–Uhlenbeck $2n$th-order oscillator. Rewrite equation (1) in the form

$$\prod_{i=1}^{n}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega_i^2\right)x = \sum_{j=0}^{n}\sigma_j^n\frac{\mathrm{d}^{2(n-j)}x}{\mathrm{d}t^{2(n-j)}} = 0,$$

where σ_j^n is the jth degree elementary symmetric polynomials in n variables $\omega_i^2, i = 1, \ldots, n$

$$\sigma_j^n = \sum_{1\leqslant i_1<\cdots<i_j\leqslant n}\omega_{i_1}^2\omega_{i_2}^2\cdots\omega_{i_j}^2, \qquad \sigma_0^n = 1, \quad 0\leqslant j\leqslant n.$$

In these notation equation (1) is equivalent to the following system of $2n$ first-order differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_{i+1}, \qquad i = 1, \ldots, 2n-1, \qquad \frac{\mathrm{d}x_{2n}}{\mathrm{d}t} = -\sum_{j=1}^{n}\sigma_j^n x_{2(n-j)+1}, \qquad x_1 = x. \tag{33}$$

System (33) has n integrals of motion. In the Pais–Uhlenbeck variables q_i, p_i

$$q_i = \sum_{j=0}^{n-1}\sigma_j^{n-1}(\hat{i})x_{2(n-j)-1}, \qquad p_i = \sum_{j=0}^{n-1}\sigma_j^{n-1}(\hat{i})x_{2(n-j)}, \qquad i = 1, \ldots, n, \tag{34}$$

where $\sigma_j^{n-1}(\hat{i})$ is the jth degree elementary symmetric polynomials in $n-1$ variables $\omega_k^2, k = 1, \ldots, \hat{i}, \ldots, n$ (the variable ω_i^2 is omitted), these integrals of motion take the form of the harmonic oscillator energy

$$H_i = \tfrac{1}{2}\left(p_i^2 + \omega_i^2 q_i^2\right).$$

In degenerate cases when some of the frequencies from the set Ω coincide, related integrals H_i coincide too. Hence, as in the case of the fourth-order oscillator, the role of a Hamiltonian which generates the dynamical equations (33) must play an appropriate linear combination of H_i. We put

$$H = \sum_{i=1}^{n} b_i H_i, \qquad b_i = \frac{1}{\omega_i\prod_{j=1,\hat{i}}^{n}\left(\omega_i^2-\omega_j^2\right)}, \tag{35}$$

where the factor with $j = i$ in the product is omitted. This Hamiltonian together with the Poisson structure Π defined by the following nonzero components

$$\Pi^{i,i+1+2j} = (-1)^j\tau_{2i-1+2j}, \qquad \tau_k = 2\sum_{i=1}^{n}\omega_i^k,$$
$$i = 1, \ldots, 2n-1, \qquad 0\leqslant j\leqslant\left[\frac{2n-i-1}{2}\right] \tag{36}$$

generates the dynamical equations (33). In formula (36) $[z]$ denote an integral part of a number z.

The quantization of the Pais–Uhlenbeck $2n$th-order oscillator we will realize by the above scheme. Assume that commutation relations among the operators $\hat{x}_i$, $i = 1, \ldots, 2n$, related to dynamical variables have the form

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\hbar \Pi^{ij}. \tag{37}$$

Using these relations and the quantum Hamiltonian $\hat{H}$ obtained from the classical one (35) by the substitution $x_i \to \hat{x}_i$, one can check that the quantum version of the dynamical equations (33)

$$\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} = \hat{x}_{i+1}, \qquad i = 1, \ldots, 2n-1, \qquad \frac{\mathrm{d}\hat{x}_{2n}}{\mathrm{d}t} = -\sum_{j=1}^{n} \sigma_j^n \hat{x}_{2(n-j)+1}$$

has the Heisenberg form (24). Using the operator solution of these equations

$$\hat{x}_1 = \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\omega_i t} a_i + \mathrm{h.c.}, \qquad \hat{x}_i = \frac{\mathrm{d}\hat{x}_{i-1}}{\mathrm{d}t}, \qquad i = 1, \ldots, 2n, \tag{38}$$

and the relations (37) we obtain the commutation relations among the creation and annihilation operators a_i, a_i^+

$$\left[a_i, a_j^+\right] = \delta_{ij}, \qquad [a_i, a_j] = \left[a_i^+, a_j^+\right] = 0, \qquad i, j = 1, \ldots, 2n. \tag{39}$$

The substitution of the solution of (38) into the Hamiltonian $\hat{H}$ gives us the Hamiltonian of the n-dimensional anisotropic oscillator

$$\hat{H} = \hbar \sum_{i=1}^{n} \omega_i \left(a_i^+ a_i + \frac{1}{2} \right). \tag{40}$$

4. Conclusion

Let us conclude these remarks by some notes on the degenerate case of the fourth-order oscillator. As was pointed out above, under the condition $\omega_1 = \omega_2 = \omega$ it is convenient to exploit the independent integrals (12) which take form

$$\begin{aligned} C_{s1} &= \tfrac{1}{2}x_4^2 + \omega^2 x_3^2 + \omega^4 x_1 x_3 - \tfrac{1}{2}\omega^4 x_2^2, \\ C_{s2} &= -\tfrac{1}{2}x_3^2 + \omega^2 x_2^2 + \tfrac{1}{2}\omega^4 x_1^2 + x_2 x_4. \end{aligned} \tag{41}$$

Substituting $\omega_1 = \omega_2 = \omega$ into formulae (22) we obtain the following commutation relations:

$$\begin{aligned} &[\hat{x}_1, \hat{x}_2] = \mathrm{i}\hbar f, && [\hat{x}_1, \hat{x}_4] = \mathrm{i}\hbar g, \\ &[\hat{x}_2, \hat{x}_3] = -\mathrm{i}\hbar g, && [\hat{x}_3, \hat{x}_4] = -\mathrm{i}\hbar\left(\omega^4 f + 2\omega^2 g\right). \end{aligned} \tag{42}$$

Using these relations and the solution of the quantum dynamical equations (23) in the degenerate case

$$\begin{aligned} \hat{x}_1 &= \mathrm{e}^{-\mathrm{i}\omega t} a_1 + t\, \mathrm{e}^{-\mathrm{i}\omega t} a_2 + \mathrm{h.c.}, \\ \hat{x}_2 &= -\mathrm{i}\omega\, \mathrm{e}^{-\mathrm{i}\omega t} a_1 + (1 - \mathrm{i}\omega t)\, \mathrm{e}^{-\mathrm{i}\omega t} a_2 + \mathrm{h.c.}, \\ \hat{x}_3 &= -\omega^2\, \mathrm{e}^{-\mathrm{i}\omega_1 t} a_1 - \omega(2\mathrm{i} + \omega t)\, \mathrm{e}^{-\mathrm{i}\omega t} a_2 + \mathrm{h.c.}, \\ \hat{x}_4 &= \mathrm{i}\omega^3\, \mathrm{e}^{-\mathrm{i}\omega_1 t} a_1 + \omega^2(-3 + \mathrm{i}t\omega)\, \mathrm{e}^{-\mathrm{i}\omega t} a_2 + \mathrm{h.c.} \end{aligned} \tag{43}$$

we obtain the commutation relations among the creation and annihilation operators a_i, a_i^+

$$[a_1, a_1^+] = \frac{\hbar}{4}\frac{3\omega^2 f + g}{\omega^3}, \qquad [a_2, a_2^+] = 0, \qquad [a_1, a_2^+] = [a_1^+, a_2] = -\frac{\mathrm{i}\hbar}{4}\frac{\omega^2 f + g}{\omega^2}. \qquad (44)$$

We remark that the pair a_2, a_2^+ commutes for any parameters f, g. It is useful to fix these parameters by the conditions

$$[a_1, a_1^+] = 1, \qquad [a_2, a_2^+] = [a_1, a_2^+] = [a_2, a_1^+] = 0.$$

We obtain $g = -\omega^2 f$ and $f = \frac{2\omega}{\hbar}$. The integrals of motions (41) in the terms of a_i^+, a_i have the form

$$C_{s1} = 16\omega^4 a_2^+ a_2 + 4\mathrm{i}\omega^5 (a_2 a_1^+ - a_1 a_2^+), \qquad C_{s2} = -8\omega^4 a_2^+ a_2 - 4\mathrm{i}\omega^3 (a_2 a_1^+ - a_1 a_2^+).$$

Commutativity of the operators a_2, a_2^+ tells us that constructing the state space of this degenerate Pais–Uhlenbeck oscillator it is necessary to take into account their classical character.

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